Proximal Operator of Quotient Functions with Application to a Feasibility Problem in Query Optimization

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Abstract

In this paper we determine the proximity functions of the sum and the maximum of componentwise (reciprocal) quotients of positive vectors. For the sum of quotients, denoted by Q_1, the proximity function is just a componentwise shrinkage function which we call q-shrinkage. This is similar to the proximity function of the ℓ_1-norm which is given by componentwise soft shrinkage. For the maximum of quotients Q_∞, the proximal function can be computed by first order primal dual methods involving epigraphical projections.

The proximity functions of Q_ν, $\nu = 1, \infty$ are applied to solve convex problems of the form $\operatorname{argmin}_x Q_\nu(\frac{Ax}{b})$ subject to $x \geq 0$, $\mathbf{1}^\top x \leq 1$. Such problems are of interest in selectivity estimation for cost-based query optimizers in database management systems.

1 Introduction

This work is motivated by query optimization in database management systems (DBMSs) where the optimal query execution plan depends on the accurate estimation of the proportion of tuples, called selectivities, that satisfy the predicates in the query. Models for selectivity estimation as those in [26] require the solution of a feasibility problem. More precisely, based on an under-determined linear system of equations $Ax = b$ which has no nonnegative solution $x \geq 0$ we are looking for a 'correct' right-hand side $\hat{b}$ such that a nonnegative solution exists. There exists a large amount of literature on feasibility problems, see [7,8] and the references therein. In particular we refer to the SMART algorithm connection with minimizing the Shannon entropy [4, 32]. However, our approach is different from the known ones with respect to the functional which has to be minimized. By the results in [28] there is a strong evidence that in query optimization it is the (reciprocal) quotients of the components $\max\{\frac{\hat{b}_i}{b_i}, \frac{b_i}{\hat{b}_i}\}$ which should be made small, not their differences. In this paper we are interested in the sum of such quotients denoted by Q_1 and their maximum Q_∞.

Recently, first order primal dual methods were successfully applied in data processing, see, e.g., the overview papers [2, 10] and the references therein. These methods are based on splitting methods known in optimization theory for a long time. In this paper we are interested in applying first order primal dual methods as an alternative to second order

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cone programming for solving problems involving the quotient functionals Q_ν, $\nu = 1, \infty$. Basically, these iterative algorithms decouple the problem into different proximation problems and the success of the method depends on the efficient solution of these proximation problems. Therefore, we examine the proximity function of our quotient functionals Q_ν, $\nu = 1, \infty$ first. We show that the proximity function of the sum of quotients, Q_1, is a componentwise shrinkage function which we call q shrinkage. It is slightly more involved than the componentwise soft-shrinkage which is the proximity function of the ℓ_1-norm since one has to solve a third order equation. The proximity function of the maximum of quotients Q_∞ can be computed by an alternating minimization method of multipliers which involves componentwise epigraphical projections. These componentwise steps can be computed in parallel.

We apply our findings to solve the feasibility problem described above and demonstrate the results obtained by different error measures by a numerical example.

The outline of this paper is as follows: In Section 2 we introduce the quotient distance between positive numbers and use it to define quotient functionals of vectors with positive components. In Section 3 we determine the proximity operator of the quotient functionals. We use our findings in Section 4 for solving feasibility problems appearing, e.g., in selectivity estimations which are necessary for query optimization in DBMSs. We describe the selectivity estimation problem, propose primal dual minimizaton algorithms and demonstrate the performance by a numerical example. Conclusions are drawn in Section 5.

2 Quotient Functions

The function $q\colon (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ defined by

$$q(x,y) := \frac{\max(x,y)}{\min(x,y)},$$

can be considered as a 'distance function'. It is symmetric in its components and since

$$q(x,y) - 1 = \frac{|x-y|}{\min(x,y)},$$

it fulfills $q(x,y) - 1 = 0$ if and only if $x = y$. Clearly, the quotient distance does not fulfill a triangle inequality. A relative of $q(x,y)$, the so-called *generalized relative distance*, given for $(x,y) \in \mathbb{R}^* \times \mathbb{R}^*$ by

$$\frac{|x-y|}{\max(x,y)},$$

has been used in [13, 19, 27, 37]. For a relation between the generalized relative error and the quotient distance we refer to [34]. Due to its *zero-homogeneity*

$$q(\lambda x, \lambda y) = q(x,y), \quad \lambda > 0 \tag{1}$$

the quotient distance is used as a contrast measure in image processing [29].

For fixed $b > 0$, we generalize $q(\cdot, b)$ to the whole real axis by $q(\cdot, b)\colon \mathbb{R} \to [0,+\infty]$ with

$$q(x,b) := \begin{cases} \frac{x}{b} & \text{if } b \le x, \\ \frac{b}{x} & \text{if } 0 < x < b, \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

The function $q(\cdot, b)$ is convex and continuous. Moreover, we have by (1) that $q(x, b) = q(x/b, 1)$. We will write just q instead of $q(\cdot, 1)$. Note that for positive arguments the function $\log q(\cdot, b) = |\log b - \log(\cdot)|$ is neither convex nor concave.

In the following, set $\mathcal{I}_N := \{1, \dots, N\}$. For fixed $b = (b_k)_{k=1}^N \in (0, +\infty)^N$ we are interested in the quotient functionals $Q_1(\cdot, b), Q_\infty(\cdot, b) : \mathbb{R}^N \to [0, +\infty]$ defined by

$$Q_1(x, b) := \sum_{k=1}^{N} q(x_k, b_k) \quad \text{and} \quad Q_\infty(x, b) := \max_{k \in \mathbb{I}_N} q(x_k, b_k). \tag{3}$$

We set $Q_\nu := Q_\nu(\cdot, 1)$, $\nu \in \{1, \infty\}$. In the following, norms $\|\cdot\|$ are Euclidean norms.

3 Proximity Operator of Quotient Functionals

Let $\Gamma_0(\mathbb{R}^N)$ denote the space of proper, convex and lower semi-continuous functions on $\mathbb{R}^N$ mapping to $\mathbb{R} \cup \{+\infty\}$. For a function $\varphi \in \Gamma_0(\mathbb{R}^N)$ and $\gamma > 0$, the *proximal function* $\operatorname{prox}_{\gamma\varphi} : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$\operatorname{prox}_{\gamma\varphi}(x) := \operatorname*{argmin}_{t \in \mathbb{R}^N} \ \varphi(t) + \frac{1}{2\gamma}\|x - t\|^2.$$

An overview of applications of proximity functions is given in [30]. For example, the proximal function of the univariate function $\varphi := |\cdot|$ is given by the so-called *soft shrinkage function with threshold* γ, i.e.,

$$\operatorname{prox}_{\gamma|\cdot|}(x) = \operatorname{soft}_\gamma(x) := \begin{cases} x - \gamma & \text{if} \quad x > \gamma, \\ 0 & \text{if} \quad x \in [-\gamma, \gamma], \\ x + \gamma & \text{if} \quad x < -\gamma. \end{cases}$$

More general the following decomposition rule holds true.

Proposition 3.1. [9, Prop. 3.6] *Let $\phi = \psi + \gamma|\cdot|$, where $\psi \in \Gamma_0(\mathbb{R})$ is differentiable at 0 with $\psi'(0) = 0$. Then $\operatorname{prox}_\phi = \operatorname{prox}_\psi \circ \operatorname{soft}_\gamma$.*

In the following we are interested in the proximal functions of $Q_1(\cdot, b)$ and $Q_\infty(\cdot, b)$. By (1) we have for $\nu \in \{1, \infty\}$ that

$$\begin{aligned} \operatorname{prox}_{\gamma Q_\nu(\cdot, b)}(x) &= \operatorname*{argmin}_{t \in \mathbb{R}^N} \ Q_\nu(t, b) + \frac{1}{2\gamma}\|x - t\|^2 \\ &= \operatorname*{argmin}_{t \in \mathbb{R}^N} \ Q_\nu\left(\frac{t}{b}, 1\right) + \frac{1}{2\gamma}\|x - t\|^2 \\ &= b \operatorname*{argmin}_{y \in \mathbb{R}^N} \ Q_\nu(y, 1) + \frac{b^2}{2\gamma}\left\|\frac{x}{b} - y\right\|^2 \\ &= b \operatorname{prox}_{\frac{\gamma}{b^2} Q_\nu}\left(\frac{x}{b}\right). \end{aligned} \tag{4}$$

Therefore it remains to consider for $\gamma > 0$ the proximal functions

$$\operatorname{prox}_{\gamma Q_\nu}(x) = \operatorname*{argmin}_{t \in \mathbb{R}^N} \ Q_\nu(t) + \frac{1}{2\gamma}\|x - t\|^2, \quad \nu \in \{1, \infty\}. \tag{5}$$

3.1 Proximity Operator of Q_1

For $\nu = 1$ the minimizer of (5) can be computed componentwise, i.e.,

$$\operatorname{prox}_{\gamma Q_1}(x) = \big(\operatorname{prox}_{\gamma q}(x_k)\big)_{k=1}^{N}. \tag{6}$$

Therefore we only have to find

$$\operatorname{prox}_{\gamma q}(x) = \operatorname*{argmin}_{t\in\mathbb{R}} \; q(t) + \frac{1}{2\gamma}(x-t)^2. \tag{7}$$

The proximal function of γq is given in the following proposition.

Proposition 3.2. *For every $\gamma > 0$ and $x \in \mathbb{R}$, we have*

$$\operatorname{prox}_{\gamma q}(x) = \begin{cases} x-\gamma & \text{if } x > 1+\gamma, \\ 1 & \text{if } x \in [1-\gamma, 1+\gamma], \\ \zeta^* \in (0,1] & \text{if } x < 1-\gamma, \end{cases} \tag{8}$$

where ζ^* is the unique solution of $\zeta^3 - x\zeta^2 - \gamma = 0$ in $(0,1)$.

Proof. To apply Proposition 3.1 we decompose q as

$$q(t) = 1 + \phi(t-1), \tag{9}$$

where $\phi := \psi + |\cdot|$ and

$$\psi(t) := \begin{cases} 0 & \text{if } t \geq 0, \\ t + \dfrac{1}{1+t} - 1 & \text{if } t \in (-1,0), \\ +\infty & \text{otherwise.} \end{cases}$$

The function ψ is in $\Gamma_0(\mathbb{R})$, it is differentiable at zero and $\psi'(0) = 0$. We want to find the proximal function of $\gamma\psi$, $\gamma > 0$. Clearly, we have for $x \geq 0$ that $\operatorname{prox}_{\gamma\psi}(x) = x$. For $x < 0$ we obtain

$$\operatorname{prox}_{\gamma\psi}(x) = \operatorname*{argmin}_{t\in(-1,0)} \; \psi(t) + \frac{1}{2\gamma}(t-x)^2.$$

The minimizer is the zero of the derivative of the objective function, i.e. has to fulfill

$$\begin{aligned} 0 &= 1 - \frac{1}{(1+t)^2} + \frac{1}{\gamma}(t-x), \\ 0 &= (1+t)^3 - (x+1-\gamma)(1+t)^2 - \gamma. \end{aligned} \tag{10}$$

In summary we have

$$\operatorname{prox}_{\gamma\psi}(x) = \begin{cases} x & \text{if } x \geq 0, \\ t^* & \text{otherwise,} \end{cases}$$

where $t^* \in (-1,0)$ is the solution of (10). By Proposition 3.1 we conclude that

$$\operatorname{prox}_{\gamma\phi}(x) = \operatorname{prox}_{\gamma\psi} \circ \operatorname{soft}_{\gamma} \tag{11}$$

$$= \begin{cases} x-\gamma & \text{if } x > \gamma, \\ 0 & \text{if } x \in [-\gamma, \gamma], \\ t^* & \text{if } x < -\gamma, \end{cases} \tag{12}$$

where t^* is the solution of

$$(1+t)^3 - (x+1)(1+t)^2 - \gamma = 0. \tag{13}$$

Finally, we obtain by (9) that

$$\begin{aligned}
\operatorname{prox}_{\gamma q}(x) &= \operatorname*{argmin}_{t\in\mathbb{R}} \; 1 + \phi(t-1) + \frac{1}{2\gamma}(t-x)^2 \\
&= 1 + \operatorname*{argmin}_{y\in\mathbb{R}} \; \phi(y) + \frac{1}{2\gamma}(y-(x-1))^2 \\
&= 1 + \operatorname{prox}_{\gamma\phi}(x-1) \\
&= \begin{cases} x-\gamma & \text{if } x > 1+\gamma, \\ 1 & \text{if } x \in [1-\gamma, 1+\gamma], \\ 1+t^* & \text{if } x < 1-\gamma, \end{cases}
\end{aligned} \tag{14}$$

where t^* is the unique solution in $(-1,0)$ of

$$(1+t)^3 - (1+t)^2 x - \gamma = 0,$$

see Remark 3.1. Setting $\zeta := 1+t$ we obtain (8). □

Remark 3.1. *We ask for the positive zeros of the polynomial $P(t) = t^3 - xt^2 - \gamma$, where $x < 1-\gamma$, $\gamma > 0$. We have*

$$P'(t) = t(3t-2x), \quad P''(t) = 6t - 2x$$

and $P(0) = -\gamma < 0$, $P(1) = 1 - x - \gamma > 0$. We distinguish two cases.

Case 1. *Let $x \le 0$. Then P is strictly convex and strictly monotone increasing on $(0,1)$. Consequently it has a unique zero t^* in $(0,1)$.*

Case 2. *Let $0 < x < 1-\gamma$. Then P is strictly convex and strictly monotone increasing on $\left(\frac{2x}{3}, 1\right)$ and $P\left(\frac{2x}{3}\right) = -\frac{4}{27}x^3 - \gamma < 0$. Hence P has a unique zero t^* in $\left(\frac{2x}{3}, 1\right)$. Further we have for the three zeros $z_1 = t^* > 0, z_2, z_3$ of P that z_2, z_3 are either conjugate complex or both of them are negative or positive since $z_1 z_2 z_3 = \gamma$. The later is not possible since $z_1 z_2 + z_1 z_3 + z_2 z_3 = 0$. Therefore P has exactly one non-negative, real-valued zero.*

In summary the zero $t^* \in (0,1)$ of the polynomial P can be computed by Newton's method with starting point $t^{(0)} = 1$, where the convergence is monotone and quadratic by the convexity and strict monotonicity of P in $(t^*, 1)$, see [20]. Alternatively we could use Cardan's formula for computing the zero.

We call $\operatorname{prox}_{\gamma q}(x)$ *q-shrinkage of x with threshold γ*. The function is illustrated in Fig. 1.

Remark 3.2. (Proximity operator of $q(\cdot, b)$, for $b > 0$)
By (4) we obtain

$$\operatorname{prox}_{\gamma q(\cdot,b)}(x) = \begin{cases} x - \frac{\gamma}{b} & \text{if } x > b + \frac{\gamma}{b}, \\ b & \text{if } x \in [b - \frac{\gamma}{b}, b + \frac{\gamma}{b}], \\ \zeta^* & \text{if } x < b - \frac{\gamma}{b}, \end{cases}$$

where $\zeta^* \in (0, b]$ is the unique positive solution of $\zeta^3 - x\zeta^2 - \gamma b = 0$.

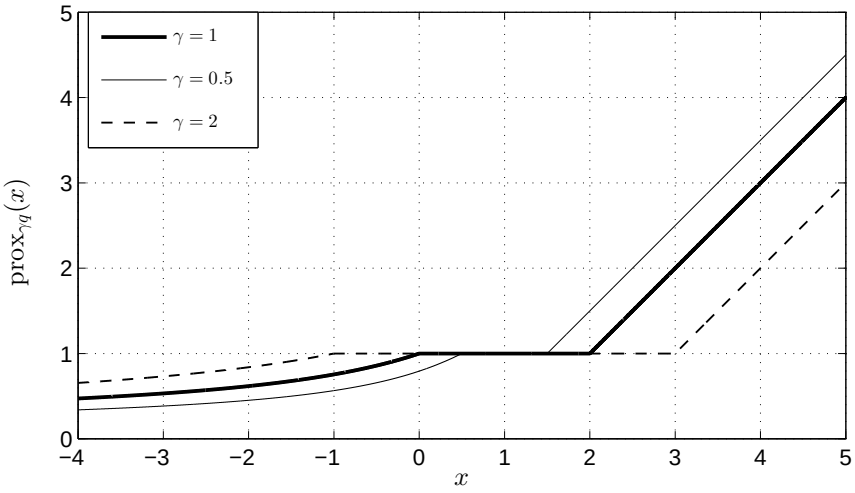

Figure 1: q-shrinkage with different thresholds γ.

3.2 Proximity Operator of Q_∞

Next we want to compute

$$\mathrm{prox}_{\gamma Q_\infty}(x) = \underset{t\in\mathbb{R}^N}{\mathrm{argmin}} \; Q_\infty(t) + \frac{1}{2\gamma}\|x-t\|^2.$$

We can treat this minimization problem as an epigraphical constraint minimization problem. Such problems were considered for example in [6, 21]. Recall that the *epigraph* of a function $\varphi : \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ is the set

$$\mathrm{epi}\,\varphi := \{(x,\xi)\in\mathbb{R}^N\times\mathbb{R} : \varphi(x)\le\xi\}.$$

If $\varphi\in\Gamma_0(\mathbb{R}^N)$, then $\mathrm{epi}\,\varphi$ is a non-empty, closed, convex set. Having this definition in mind, our minimization problem can be rewritten as

$$\mathrm{prox}_{\gamma Q_\infty}(x) = \underset{t=(t_k)_{k=1}^N\in\mathbb{R}^N,\xi\in\mathbb{R}}{\mathrm{argmin}} \; \xi + \frac{1}{2\gamma}\|x-t\|^2 \quad \text{s.t. } ((t_k,\xi))_{k=1}^N \in (\mathrm{epi}\,q)^N. \tag{15}$$

Using the indicator function of a set $\Omega\subset\mathbb{R}^M$, where $M\in\mathbb{N}^*$, defined by

$$\iota_\Omega(x) := \begin{cases} 0 & \text{if } x\in\Omega, \\ +\infty & \text{otherwise}, \end{cases}$$

we see that (15) can be further rewritten as

$$\mathrm{prox}_{\gamma Q_\infty}(x) = \underset{\substack{(t,\xi)\in\mathbb{R}^{N+1}\\(s,\eta)\in\mathbb{R}^{2N}}}{\mathrm{argmin}} \; \xi + \frac{1}{2\gamma}\|x-t\|^2 + \sum_{k=1}^N \iota_{\mathrm{epi}\,q}(s_k,\eta_k) \quad \text{s.t. } s=t,\ \xi\mathbf{1}_N=\eta, \tag{16}$$

where $\mathbf{1}_N$ denotes the vector consisting of N entries 1. This problem can be solved e.g. by an alternating direction method of multipiers (ADMM) algorithm [10, 16, 17] as follows:

Algorithm 1 ADMM for $\operatorname{prox}_{\gamma Q_\infty}$

Initialization: $\mu > 0$

$\begin{pmatrix} s^{(0)} \\ \eta^{(0)} \end{pmatrix} \in \mathbb{R}^{2N}$ and $\begin{pmatrix} p_t^{(0)} \\ p_\xi^{(0)} \end{pmatrix} \in \mathbb{R}^{2N}$.

Iterations:

For $r = 0, 1, \dots$

1. $\begin{pmatrix} t^{(r+1)} \\ \xi^{(r+1)} \end{pmatrix} = \underset{(t,\xi)\in\mathbb{R}^{N+1}}{\operatorname{argmin}} \ \xi + \frac{1}{2\gamma}\|x - t\|^2 + \frac{\mu}{2}\big(\|t - s^{(r)} + p_t^{(r)}\|^2 + \|\xi \mathbf{1}_N - \eta^{(r)} + p_\xi^{(r)}\|^2\big),$
2. $\begin{pmatrix} s^{(r+1)} \\ \eta^{(r+1)} \end{pmatrix} = \underset{(s,\eta)\in\mathbb{R}^{2N}}{\operatorname{argmin}} \ \sum_{k=1}^{N} \iota_{\operatorname{epi} q}(s_k, \eta_k) + \frac{\mu}{2}\big(\|t^{(r+1)} - s + p_t^{(r)}\|^2 + \|\xi^{(r+1)} \mathbf{1}_N - \eta + p_\xi^{(r)}\|^2\big),$
3. $\begin{pmatrix} p_t^{(r+1)} \\ p_\xi^{(r+1)} \end{pmatrix} = \begin{pmatrix} p_t^{(r)} \\ p_\xi^{(r)} \end{pmatrix} + \begin{pmatrix} t^{(r+1)} \\ \xi^{(r+1)} \mathbf{1}_N \end{pmatrix} - \begin{pmatrix} s^{(r+1)} \\ \eta^{(r+1)} \end{pmatrix}.$

The sequence $(t^{(r)}, \xi^{(r)})_{r\in\mathbb{N}}$ generated by the ADMM algorithm is ensured to converge to a solution of problem (16) by [10].

The minimizer in the first step can be computed separately for t and ξ. Setting the gradients of the corresponding functionals to zero we obtain

$$t^{(r+1)} = \frac{1}{1+\gamma\mu}\big(x + \mu\gamma(s^{(r)} - p_t^{(r)})\big) \quad \text{and} \quad \xi^{(r+1)} = \frac{1}{N}\left(\sum_{k=1}^{N}(\eta_k^{(r)} - p_{\xi,k}^{(r)}) - \frac{1}{\mu}\right).$$

The second proximity problem can be solved separately for $k \in \mathcal{I}_N$. For each component it requires the projection of $(t_k^{(r+1)} + p_{t,k}^{(r)}, \xi^{(r+1)} + p_{\xi,k}^{(r)})$ onto $\operatorname{epi} q$. The projection onto $\operatorname{epi} q$ is considered in the next proposition:

Proposition 3.3. *The projection $\mathcal{P}_{\operatorname{epi} q}(u, \zeta)$ of $(u, \zeta) \in \mathbb{R}^2$ onto the epigraph of q is given by*

$$\mathcal{P}_{\operatorname{epi} q}(u,\zeta) := \begin{cases} (u,\zeta) & \text{if } u > 0 \ \wedge \ \max\{u, \frac{1}{u}\} \le \zeta, \\ (\frac{1}{2}(u+\zeta), \frac{1}{2}(u+\zeta)) & \text{if } 2 - u < \zeta < u, \\ (1,1) & \text{if } \zeta \le \min\{2-u, u\}, \\ (t^*, \frac{1}{t^*}) & \text{if } u < \zeta \ \wedge \ \zeta < \frac{1}{u} \text{ if } u > 0, \end{cases} \tag{17}$$

where t^* is the solution of the fourth order equation $P(t) := t^4 - ut^3 + \zeta t - 1 = 0$ in $(0, 1)$.

Proof. The points in the different areas

$$A_1 := \{(u,\zeta) : u > 0 \ \wedge \ \max\{u, \tfrac{1}{u}\} \le \zeta\}, \quad A_2 := \{(u,\zeta) : 2 - u < \zeta < u\},$$
$$A_3 := \{(u,\zeta) : \zeta \le \min\{2-u, u\}\}, \quad A_4 := \{(u,\zeta) : u < \zeta \ \wedge \ \zeta < \tfrac{1}{u} \text{ if } u > 0\},$$

depicted in Fig. 2 are projected in different ways.

The points in A_1 are already in $\operatorname{epi} q$ and were therefore mapped to themselves. The points in the normal cone A_3 of $\operatorname{epi} q$ at $(1, 1)$ are obviously projected to $(1, 1)$. For $(u, \zeta) \in A_2$ the orthogonal projection $(t, \theta) = \mathcal{P}_{\operatorname{epi} q}(u, \zeta)$ has to fulfill $\theta = t$ and

$$\left\langle \begin{pmatrix} u \\ \zeta \end{pmatrix} - \begin{pmatrix} t \\ \theta \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = 0,$$

which results in $t = \frac{1}{2}(u+\zeta)$. Finally, the points in A_4 are projected onto the curve $\tau(t) := (t, 1/t)$, $t \in (0, 1)$. This curve has the tangent vectors $(1, -1/t^2)$. Thus, $(t, \theta) = \mathcal{P}_{\operatorname{epi} q}(u, \zeta)$ has to satisfy $\theta = 1/t$ and

$$\left\langle \begin{pmatrix} u - t \\ \zeta - \frac{1}{t} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{t^2} \end{pmatrix} \right\rangle = 0,$$

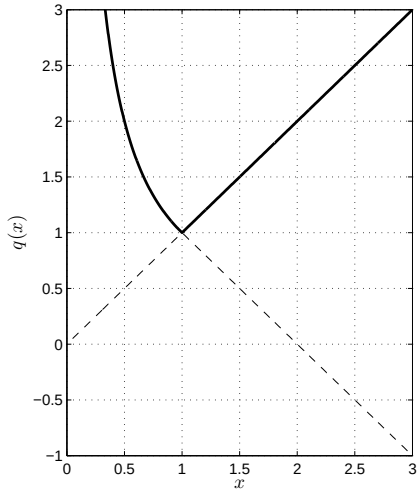

Figure 2: Areas for the epigraphical projection onto epi_q.

which leads to

$$t^4 - ut^3 + \zeta t - 1 = 0.$$

For the zeros of the above polynomial see the next remark. □

Remark 3.3. *Consider $P(t) = t^4 - ut^3 + \zeta t - 1$, where $u < \zeta$ and $u\zeta < 1$ if $u > 0$. We have*

$$P'(t) = 4t^3 - 3ut^2 + \zeta, \quad P''(t) = 6t\,(2t - u).$$

Let $t_0 := \max\big(\frac{u}{2}, 0\big)$. Then $P''(t) > 0$ for $t \in (t_0, 1)$ so that P is strictly convex and P' is strictly monotone increasing on $(t_0, 1)$. Further we conclude $P(0) = -1 < 0$ and $P(1) = \zeta - u > 0$. According to the sign of u we distinguish the two cases.

Case 1. *Let $u \leq 0$. Then $P(t_0) = P(0) < 0$ and the convexity of P implies that P has exactly one zero t^* in $[0, 1]$ and is strictly monotone increasing on $(t^*, 1)$.*

Case 2. *Let $0 < u < \zeta$. Then $u\zeta < 1$ implies that $u < 1$ and thus $\frac{u}{2} \in (0, \frac{1}{2})$. Now $P(t_0) = P(\frac{u}{2}) = -\frac{u^4}{16} + \frac{1}{2}\big(\zeta u - 1\big) - \frac{1}{2} < 0$ and it follows again that P has exactly one zero t^* in $\big[\frac{u}{2}, 1\big]$ and is strictly monotone increasing on $(t^*, 1)$. Straightforward computation shows that P is strictly concave on $\big(0, \frac{u}{2}\big)$ and $P'\left(\frac{u}{2}\right) = -\frac{u^3}{4} + \zeta > 0$, $P'(0) > 0$. Consequently P has no zero in $\big[0, \frac{u}{2}\big]$.*

In summary P has exactly one zero t^* in $(0, 1)$. Since P is convex and strictly increasing on $[t^*, 1]$, the iterates of Newton's algorithm with starting point $t^{(0)} = 1$ converge monotonically to t^* with quadratic convergence rate, see [20].

Finally, we want give also an analytic solution of $P(t) = 0$. Setting $t = x + \dfrac{u}{4}$, we can convert this quartic equation into a depressed quartic equation, and we obtain

$$\tilde{P}(x) := P(x + \frac{u}{4}) = x^4 + \alpha x^2 + \beta x + \gamma. \tag{18}$$

Then we can apply *Ferrari's method* [35] or *Lagrange's method* [24] to solve $\tilde{P}(x) = 0$ as shown in the following lemma.

Lemma 3.1. *The solutions of* $P(t) = 0$ *are* $(t_1, t_2, t_3, t_4) \in \mathbb{C}^4$ *such that for every* $i \in \{1, 2, 3, 4\}$, $t_i = x_i + \dfrac{u}{4}$, *where* $(x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ *are the solutions of* $\tilde{P}(x) = 0$ *given by*

$$\begin{cases} x_1 = \frac{1}{2}\left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right), \\ x_2 = \frac{1}{2}\left(\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}\right), \\ x_3 = \frac{1}{2}\left(-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}\right), \\ x_4 = \frac{1}{2}\left(-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}\right), \end{cases} \tag{19}$$

where $(z_1, z_2, z_3) \in \mathbb{C}^3$ *are solutions of the third order equation*

$$R(z) := z^3 + 2\alpha z^2 + (\alpha^2 - 4\gamma)z - \beta^2 = 0, \tag{20}$$

with

$$\alpha := -\frac{3u^2}{8}, \quad \beta := \zeta - \frac{u^3}{8}, \quad \gamma := \frac{\zeta u}{4} - \frac{3u^4}{4^4} - 1.$$

The zeros of the third order polynomial R can be found using *Cardan's method* detailed in the appendix. The following proposition gives the epigraphical projection onto $\operatorname{epi} q(\cdot, b)$, for $b > 0$:

Proposition 3.4. *The epigraphical projection at* $(u, \zeta) \in \mathbb{R}^2$ *onto* $\operatorname{epi} q(\cdot, b)$ *for fixed* $b > 0$ *is given by*

$$\mathcal{P}_{\operatorname{epi} q(\cdot,b)}(u, \zeta) := \begin{cases} (u, \zeta) & \text{if } u > 0 \ \wedge \ \max\{\frac{u}{b}, \frac{b}{u}\} \leq \zeta, \\ (\frac{b}{1+b^2}(bu + \zeta), \frac{1}{1+b^2}(bu + \zeta)) & \text{if } 1 + b^2 - bu < \zeta < \frac{u}{b}, \\ (b, 1) & \text{if } \zeta \leq \min\{1 + b^2 - bu, 1 - b^2 + bu\}, \\ (t^*, \frac{b}{t^*}) & \text{if } 1 - b^2 + bu < \zeta \ \wedge \ \left(\zeta < \frac{b}{u} \text{ if } u > 0\right), \end{cases} \tag{21}$$

where t^* *is the solution of the fourth order equation* $P(t) := t^4 - ut^3 + \zeta bt - b^2 = 0$ *in* $(0, b)$.

Proof. Let $b > 0$. Similar to the proof of Proposition 3.3, we consider the four areas A_1, A_2, A_3 and A_4. To this end, we need to find the perpendicular of $u \in (0, b] \mapsto q_1(u, b) := b/u$ and $u \in [b, +\infty) \mapsto q_2(u, b) := u/b$ at $(b, 1)$.

Let $u \in (0, b]$. The perpendicular of $q_1(u, b)$ at $(b, 1)$, denoted by $p_1(u) := \alpha_1 u + \beta_1$, where $(\alpha_1, \beta_1) \in \mathbb{R}^2$, satisfies

$$\begin{cases} \langle (u, \zeta) - (t, \theta), (1, -b/t^2) \rangle = 0 \\ (t, \theta) = (b, 1) \end{cases} \quad \Leftrightarrow \quad \langle (u, \zeta) - (b, 1), (1, -1/b) \rangle = 0$$

$$\Leftrightarrow \quad u - b - \frac{1}{b}(\zeta - 1) = 0$$

$$\Leftrightarrow \quad bu - b^2 + 1 = \zeta.$$

Thus,

$$p_1(u) := bu + 1 - b^2. \tag{22}$$

Let $u \in [b, +\infty)$. The perpendicular of $q_2(u, b)$ at $(b, 1)$, denoted by $p_2(u)$ satisfies, for $\beta_2 \in \mathbb{R}$,

$$\begin{cases} p_2(u) = -bu + \beta_2 \\ p_2(b) = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} p_2(u) = -bu + \beta_2 \\ -b^2 + \beta_2 = 1 \end{cases}$$

$$\Leftrightarrow \quad p_2(u) = -bu + 1 + b^2. \tag{23}$$

According to (22) and (23), the areas are given by

$$\begin{aligned}
A_1 &= \{(u,\zeta)\,;\, u>0 \wedge \zeta \geq \max\{u/b, b/u\}\},\\
A_2 &= \{(u,\zeta)\,;\, 1+b^2-bu<\zeta<u/b\},\\
A_3 &= \{(u,\zeta)\,;\, \zeta \leq \min\{1-b^2+bu, 1+b^2-bu\}\},\\
A_3 &= \{(u,\zeta)\,;\, \zeta > 1-b^2+bu \wedge (\zeta < b/u \text{ if } u>0)\}.
\end{aligned}$$

The points in A_1 are already in $\operatorname{epi} q(\cdot,b)$ and were therefore mapped to themselves. The points in the normal cone A_3 of $\operatorname{epi} q(\cdot,b)$ at $(b,1)$ are obviously projected to $(b,1)$. For $(u,\zeta)\in A_2$ the orthogonal projection $(t,\theta)=\mathcal{P}_{\operatorname{epi} q}(u,\zeta)$ has to fulfill $\theta = t/b$ and

$$\left\langle \begin{pmatrix} u \\ \zeta \end{pmatrix} - \begin{pmatrix} t \\ \theta \end{pmatrix}, \begin{pmatrix} 1 \\ 1/b \end{pmatrix} \right\rangle = 0,$$

which results in $t = \frac{b}{1+b^2}(bu+\zeta)$. Finally, the points in A_4 are projected onto the curve $\tau(t) := (t, b/t)$, $t\in(0,b)$. This curve has the tangent vectors $(1, -b/t^2)$. Thus, $(t,\theta) = \mathcal{P}_{\operatorname{epi} q}(u,\zeta)$ has to satisfy $\theta = b/t$ and

$$\left\langle \begin{pmatrix} u-t \\ \zeta - \frac{b}{t} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{b}{t^2} \end{pmatrix} \right\rangle = 0,$$

which leads to $t^4 - ut^3 + \zeta bt - b^2 = 0$. □

4 A Feasibility Problem in Selectivity Estimation

The aim of this section is to solve for $\nu\in\{1,\infty\}$ the minimization problem

$$\operatorname*{argmin}_{x\in\mathbb{R}^N} \; Q_\nu\left(Ax, b\right) \quad \text{subject to } x\in\triangle \tag{24}$$

where $A\in\mathbb{R}^{M\times N}$ and

$$\triangle := \{x\in[0,+\infty)^N : \sum_{k=1}^{N} x_k \leq 1\}.$$

Such problems arise, e.g., in the estimation of selectivities for cost-based query optimizers in DBMSs, see [26, 28]. A brief sketch of the selectivity estimation task is given in the following subsection.

4.1 Selectivity Estimation in DBMSs

Selectivities indicate the proportion of tupels in a database that satisfy the predicates in a query. The accurate estimation of selectivities is crucial for the design of optimal query execution plans. However, in practice we have to live with inaccurate size estimations and a natural question is how these errors influence the query plan optimization. In [28] the error propagation of wrong selectivity estimates though an accordingly optimized query was examined. It appears that the error propagation is multiplicative, see also [22]. Worst-case error bounds of the cost function of an optimal plan based on erroneous selectivities were proved in terms of the cost function of the optimal plan based on accurate selectivities and the quotient of the erroneous and the accurate selectivities. The results give strong evidence that the quotient error of selectivities should be considered superior to other error measures, in particular additive ones.

The selectivity estimation problem reads as follows: Let $\mathcal{P}_n$ denote the power set of $\mathcal{I}_n := \{1, \dots, n\}$. Assume that we are given a set $\{p_i : i \in \mathcal{I}_n\}$ of simple predicates. According to [26] we model the selectivities of conjunctive predicates as a probability distribution. For this purpose we represent the conjunctive predicates in full disjunctive normal form (DNF). For example in case $n = 3$, the predicates p_1 and $p_1 \wedge p_2$ have the DNFs

$$\begin{aligned} p_1 &= (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \vee (p_1 \wedge p_2 \wedge p_3), \\ p_1 \wedge p_2 &= (p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3). \end{aligned}$$

We identify the sets of $\mathcal{P}_n$ with the binary strings $\beta \in \{0,1\}^n$, where $\beta_i = 1$ if and only if $i \in \mathcal{I}_n$ belongs to the set. Then $\beta := \mathbf{0}_n$ corresponds to the empty set and $\beta := \mathbf{1}_n$ to the full set $\mathcal{I}_n$. Let $\mathcal{P} := \{0,1\}^n \backslash \{\mathbf{0}_n\}$. Accordingly, we index the selectivities s_β of conjunctive predicates by binary labels $\beta \in \{0,1\}^n$, where $\beta_i = 1$ if and only if predicate p_i is in the conjunction. Finally, the selectivities x_β of the clauses v in the DNFs are indexed by binary strings $\beta \in \{0,1\}^n$, where $\beta_i = 1$ if $p_i \in v$ and $\beta_i = 0$ if $\neg p_i \in v$. Using this notation we obtain in the above example

$$\begin{aligned} s_{100} &= x_{100} + x_{110} + x_{101} + x_{111}, \\ s_{110} &= x_{110} + x_{111}. \end{aligned}$$

Clearly, we have

$$s_{\mathbf{0}_n} = \sum_{\beta \in \{0,1\}^n} x_\beta = 1$$

and $x_{\mathbf{0}_n}$ appears only as a summand in s_β for $\beta = \mathbf{0}_n$. The values x_β can be interpreted as probabilities of the appearance of the corresponding clause. Of course only a small part of selectivities s_β, $\beta \in \mathcal{J} \subset \mathcal{P}$, $\#\mathcal{J} \ll 2^n$, of conjunctive predicates can be stored via multivariate statistics in a DBMS. Let $b := (s_\beta)_{\beta \in \mathcal{J}}$, $x := (x_\beta)_{\beta \in \mathcal{P}}$ and

$$A := (a_{\beta,\beta'})_{\beta \in \mathcal{J}, \beta' \in \mathcal{P}}, \quad a_{\beta,\beta'} := \begin{cases} 1 & \text{if } \beta_i = 1 \Rightarrow \beta'_i = 1 \quad \forall\, i \in \mathcal{I}_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, if all $s_\beta, \beta \in \mathcal{J}$ were known accurately, the x_β would satisfy $Ax = b$. In [26] the authors propose to estimate x_β, $\beta \in \mathcal{P}$ (and consequently all selectivities) by maximizing the entropy

$$\max_x \sum_{\beta \in \{0,1\}^n} -x_\beta \log x_\beta \quad \text{subject to} \quad Ax = b,\ x \geq 0,\ \sum_{\beta \in \{0,1\}^n} x_\beta = 1. \tag{25}$$

If this convex optimization problem is feasible it can be solved by several methods, e.g., via a Newton method applied to the dual problem, see [15, p. 222-223]. An iterative scaling method was proposed in [26].

However, in practice, inaccuracies in the stored selectivities s_β make (25) infeasible, i.e., $Ax = b$ has no solution $x \geq 0$. Penalizing the error between Ax and b by adding a further term to the entropy would require us to determine a penalizing parameter. Therefore we deal with the feasibility problem separately and seek a feasible x first. By the reasons described at the beginning of this subsection, we are looking for a small quotient error between Ax and b, i.e., we consider (24). The result can subsequently be used to solve (25) which is not addressed in this paper.

4.2 Solution of the Feasibility Problem

One possible approach to solve problem (24) is via *second order cone programming* (SOCP) as proposed, e.g., in [34]. For details on SOCP we refer, e.g., to [25]. In the following, we show how the problem can be tackled by first order primal dual algorithms. These iterative algorithms have the advantage that certain steps in each iteration as the q thresholding or the epigraphical projections can be computed in parallel. In particular, the Q_1 approach appears to be rather fast.

We rewrite the problem (24) as

$$\operatorname*{argmin}_{x\in\mathbb{R}^N, y\in\mathbb{R}^M} Q_\nu\left(y,b\right) + \iota_\triangle(x) \quad \text{subject to } Ax = y. \tag{26}$$

The above optimization problem can be solved by various primal-dual algorithms [5, 11, 12, 23, 31, 36].

For $\nu = 1$ we apply the primal-dual hybrid gradient method with an extrapolation of the dual variable (PDHGMp) from [3, 5, 14, 29] to solve (26), see appendix:

Algorithm 2 PDHGMp for (26)

Initialization: $\mu > 0$. $\sigma > 0$ with $\mu\sigma < 1/\|A\|_2^2$, $\theta \in (0,1]$
$x^{(0)}$, $p^{(0)} = \bar{p}^{(0)}$.
Iterations:
For $r = 0, 1, \ldots$

$$\begin{aligned}
&1.\ x^{(r+1)} &&= \operatorname*{argmin}_{x\in\mathbb{R}^N} \iota_\triangle(x) + \tfrac{1}{2\mu}\|x - \left(x^{(r)} - \mu\sigma A^\top \bar{p}^{(r)}\right)\|^2 \\
&2.\ y^{(r+1)} &&= \operatorname{prox}_{Q_1(\cdot,b)/\sigma}(p^{(r)} + Ax^{(r+1)}) \\
&3.\ p^{(r+1)} &&= p^{(r)} + Ax^{(r+1)} - y^{(r+1)} \\
&4.\ \bar{p}^{(r+1)} &&= p^{(r+1)} + \theta(p^{(r+1)} - p^{(r)})
\end{aligned}$$

The sequence $(x^{(r)}, y^{(r)})_{r\in\mathbb{N}}$ generated by Algorithm 2 is ensured to converge to a solution of problem (26) by [5], see also [3, Theorem 6.4].

The first step is just a projection of $x^{(r)} - \mu\sigma A^\top \bar{p}^{(r)}$ onto $\triangle$. The second step requires the solution of a proximity problem for $Q_1(\cdot, b)$ which can be done by applying componentwise the q-shrinkage described in Remark 3.2.

For $\nu = \infty$ we reformulate problem (26) as

$$\operatorname*{argmin}_{(x,\xi)\in\mathbb{R}^{N+1}, (y,\eta)\in\mathbb{R}^{2M}} \xi + \sum_{k=1}^{M} \iota_{\operatorname{epi} q(\cdot,b_k)}(y_k, \eta_k) + \iota_\triangle(x) \quad \text{s.t. } Ax = y,\ \xi \mathbf{1}_M = \eta \tag{27}$$

and apply the PDHGMp algorithm again:

Algorithm 3 PDHGMp for (27)

Initialization: $\mu > 0$. $\sigma > 0$ with $\mu\sigma < \min(1/\|A\|_2^2, 1/M)$, $\theta \in (0,1]$
$x^{(0)}$, $p^{(0)} = \bar{p}^{(0)}$.
Iterations:
For $r = 0, 1, \ldots$

$$\left\lfloor \begin{array}{rlcl} 1. & \begin{pmatrix} x^{(r+1)} \\ \xi^{(r+1)} \end{pmatrix} & = & \underset{(x,\xi)\in\mathbb{R}^{N+1}}{\operatorname{argmin}} \; \xi + \iota_{\triangle}(x) + \frac{1}{2\mu} \left\| \begin{pmatrix} x \\ \xi \end{pmatrix} - \left(\begin{pmatrix} x^{(r)} \\ \xi^{(r)} \end{pmatrix} - \mu\sigma \begin{pmatrix} A^\top \bar{p}_x^{(r)} \\ \mathbf{1}_M^\top \bar{p}_\xi^{(r)} \end{pmatrix} \right) \right\|^2 \\ 2. & \begin{pmatrix} y^{(r+1)} \\ \eta^{(r+1)} \end{pmatrix} & = & \underset{(y,\eta)\in\mathbb{R}^{2M}}{\operatorname{argmin}} \; \sum_{k=1}^{M} \iota_{\operatorname{epi} q(\cdot,b_k)}(y_k, \eta_k) + \frac{\sigma}{2} \left\| \begin{pmatrix} y - (p_x^{(r)} + Ax^{(r+1)}) \\ \eta - (p_\xi^{(r)} + \mathbf{1}_M \xi^{(r+1)}) \end{pmatrix} \right\|^2 \\ 3. & \begin{pmatrix} p_x^{(r+1)} \\ p_\xi^{(r+1)} \end{pmatrix} & = & \begin{pmatrix} p_x^{(r)} + Ax^{(r+1)} - y^{(r+1)} \\ p_\xi^{(r)} + \mathbf{1}_M \xi^{(r+1)} - \eta^{(r+1)} \end{pmatrix} \\ 4. & \begin{pmatrix} \bar{p}_x^{(r+1)} \\ \bar{p}_\xi^{(r+1)} \end{pmatrix} & = & \begin{pmatrix} p_x^{(r+1)} + \theta(p_x^{(r+1)} - p_x^{(r)}) \\ p_\xi^{(r+1)} + \theta(p_\xi^{(r+1)} - p_\xi^{(r)}) \end{pmatrix} \end{array} \right.$$

In the first step we have to compute the projection of $x^{(r)} - \mu\sigma A^\top \bar{p}_x^{(r)}$ onto the probability simplex and

$$\xi^{(r+1)} = \xi^{(r)} - \mu\sigma \mathbf{1}_M^\top \bar{p}_\xi^{(r)} - \mu.$$

The second step requires just the componentwise epigraphical projection of the tuple $((p_x^{(r)} + Ax^{(r+1)})_k, (p_\xi^{(r)} + \mathbf{1}_M \xi^{(r+1)})_k)$ onto the epigraph of $q(\cdot, b_k)$, $k = 1, \ldots, M$, which can be realized by Proposition 3.4.

Example 4.1. *We use the notation from Subsection 4.1 and consider the linear system of equations*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_{001} \\ x_{010} \\ x_{011} \\ x_{100} \\ x_{101} \\ x_{110} \\ x_{111} \end{pmatrix} = \begin{pmatrix} 0.2114 \\ 0.6331 \\ 0.6312 \\ 0.5182 \\ 0.9337 \\ 0.0035 \end{pmatrix} = \begin{pmatrix} s_{001} \\ s_{010} \\ s_{100} \\ s_{011} \\ s_{101} \\ s_{110} \end{pmatrix}$$

which has no solution $x \geq 0$. Note that the last component of the vector b on the right-hand side differs from the other ones by a magnitude of order. We solve problem (24) for $\nu = 1$ by Algorithm 2 and for $\nu = \infty$ by Algorithm 3, with stopping criterion $\|Ax - y\|_\infty < \epsilon$ and the parameters in Table 1, where $s = 1/\|A\|_2$ and $A \in \mathbb{R}^{m\times n}$. Note that we have chosen the parameters in Algorithm 2 slightly larger than $\mu\sigma\|A\|_2^2 < 1$ to get a faster convergence although the convergence has not been proved theoretically for such parameters. Step 2 of Algorithm 3 is computed using Newton's algorithm, which is faster than computing an analytic solution while providing a high precision.

For comparison we solve the following feasibility problems with the additive errors

$$\underset{x\in\mathbb{R}^N}{\operatorname{argmin}} \|Ax - b\|_p \quad \textit{subject to} \quad x \in \triangle \tag{28}$$

for $p \in \{1, 2\}$ by linear and quadratic programming routines from MOSEK [1], respectively. Having the solutions $\hat{x}$ of the four problems we compute the vectors $\hat{b} := A\hat{x}$. Note that $\hat{x}$ itself is not of interest here because it will be optimized later, e.g., in the database

application by entropy minimization. The errors $Q_\infty(\hat{b}, b)$ for the four vectors $\hat{b}$ read as follows:

method	(24) for $\nu = \infty$	(24) for $\nu = 1$	(28) for $p = 1$	(28) for $p = 2$
$Q_\infty(\hat{b}, b)$	2.61	3.65	75.65	74.85

Indeed this reflects qualitatively what we have seen in many test examples. Method (24) *with $\nu = \infty$ provides of course the minimal value $Q_\infty(\hat{b}, b)$ since the method was designed to minimize this error. However, model* (24) *with $\nu = 1$ produces only a slightly larger error $Q_\infty(\hat{b}, b)$. Since this method, which does not require epigraphical projections, is faster, it may be a good alternative choice. Finally, both methods* (28) *lead to considerably higher errors $Q_\infty(\hat{b}, b)$. From the tests we have done so far it cannot be deduced that one of this methods gives a smaller quotient error than the other one. We emphasize again that we are only interested in the Q_∞ error since this error influences the design of query execution plans.*

	μ	σ	θ	ϵ	Initialisation	It
Algorithm 2	$s/2$	$8s$	1	10^{-5}	$x^{(0)} = \frac{1}{n}\mathbf{1}_n, \quad p^{(0)} = \mathbf{1}_m$	161
Algorithm 3	$s/2$	$2s$	1	10^{-4}	$x^{(0)} = \frac{1}{n}\mathbf{1}_n, \quad p_x^{(0)} = p_\xi^{(0)} = \mathbf{1}_m$	160

Table 1: Parameters used for Example 4.1

5 Conclusions

We have determined the proximity operator of the sum and the maximum of componenwise quotient errors of positive vectors. These proximity operators may be applied in the solution of various tasks. As an example we have considered a feasibility problem appearing in the selectivity estimation for query optimization. Here we have a strong evidence that quotient distances are more relevant than additive error measures. We have proposed first order primal dual methods to solve the relevant problem and have underlined our findings by a numerical toy example. In connection with query optimization in DBMSs we are working on a GPU implementation of certain steps of the primal dual algorithms and on the solution of problem (25). We intend to give a comprehensive comparison of several methods, in particular in terms of the execution time. We have found that such a comparison is indeed a task on its own which is beyond the scope of the present paper, which explains the basic mathematical ideas.

In the future, we intend to apply quotient distances to problems appearing in image processing as for example illumination corrections based on the so-called retinex model. This model assumes that an image is given by the componentwise product of the illumination and the reflection in the scene, see [18].

6 Appendix

6.1 General PDHGMp Algorithm

For $f_1 \in \Gamma_0(\mathbb{R}^N)$, $f_2 \in \Gamma_0(\mathbb{R}^M)$ and $C \in \mathbb{R}^{M \times N}$ the solution of

$$\operatorname*{argmin}_{x,y} \; f_1(x) + f_2(y) \quad \text{subject to} \quad Cx = y$$

can be computed by the PDHGMp supposed that a saddle point of the Lagrangian $L(x,y,p) := f_1(x) + f_2(y) + \langle p, Cx - y\rangle$ exists, see [5,33].

Algorithm 4 PDHGMp

Initialization: $\mu > 0$. $\sigma > 0$ with $\mu\sigma < 1/\|C\|^2$, $\theta \in (0,1]$
$x^{(0)}$, $p^{(0)} = \bar{p}^{(0)}$.
Iterations:
For $r = 0, 1, \ldots$

$$\left\lfloor \begin{array}{lcl} 1.\ x^{(r+1)} & = & \underset{x\in\mathbb{R}^N}{\operatorname{argmin}} \quad f_1(x) + \frac{1}{2\mu}\|x - \left(x^{(r)} - \mu\sigma C^\top \bar{p}^{(r)}\right)\|^2 \\ 2.\ y^{(r+1)} & = & \underset{y\in\mathbb{R}^M}{\operatorname{argmin}} \quad f_2(y) + \frac{\sigma}{2}\|y - (p^{(r)} + Cx^{(r+1)})\|^2 \\ 3.\ p^{(r+1)} & = & p^{(r)} + Cx^{(r+1)} - y^{(r+1)} \\ 4.\ \bar{p}^{(r+1)} & = & p^{(r+1)} + \theta(p^{(r+1)} - p^{(r)}) \end{array} \right.$$

The sequence $(x^{(r)}, p^{(r)})_{r\in\mathbb{N}}$ generated by Algorithm 4 converges to a saddle point of the Lagrangian [3, Thm. 6.4].

6.2 Cardan's Formula for Solving (20)

We show how the Cardan formula can be applied for finding the zeros of the third order polynomial R in (20): After a change of variable, we have

$$\widetilde{R}(y) = R(y - \frac{2\alpha}{3}) = y^3 + py + q, \tag{29}$$

where

$$p = \alpha^2\left(1 - \frac{8\alpha}{3}\right) - 4\gamma, \quad q = -\beta^2 + \frac{8\alpha}{3}\left(\gamma - \frac{\alpha}{36}\right).$$

Solutions of equation (29) denoted by $(y_1, y_2, y_3) \in \mathbb{C}^3$ depend on the sign of the discriminant $\Delta = -\left(4p^3 + 27q^2\right)$. Then, we have the following cases:

(i) If $\Delta > 0$, then the equation has 3 distinct real roots given, for every $i \in \{1,2,3\}$, by

$$y_i = 2\sqrt{\frac{-p}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{-q}{2}\sqrt{\frac{27}{-p^3}}\right) + \frac{2(i-1)\pi}{3}\right).$$

(ii) If $\Delta = 0$, then two cases are possible: if $p = q = 0$ then the equation (29) admits 0 as a multiple root, otherwise, the equation has a multiple root and all its roots are real and equal to

$$y_1 = 2\left(-q/2\right)^{1/3} \text{ and } y_2 = y_3 = -\left(-q/2\right)^{1/3}.$$

(iii) If $\Delta < 0$, then the equation has one real root and two nonreal complex conjugate roots given by

$$y_1 = a + b, \quad y_2 = ja + \bar{j}b, \quad y_3 = j^2a + \bar{j^2}b,$$

where $a = \left(\frac{1}{2}(-q + \sqrt{-\Delta/27})\right)^{1/3}$, $b = \left(\frac{1}{2}(-q - \sqrt{-\Delta/27})\right)^{1/3}$, and $j = \mathrm{e}^{\mathrm{i}\,2\pi/3}$.

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